Hints and solutions to selected exercises

Homework 1.

Complex Numbers, polynomials

0. a) Find analytically and graphically the sum of vectors $\underline{u}$, $\underline{v}$, (the tails are in the origin of the coordinate system), their lengths and dot products

1. $\underline{u} = (3,1), \underline{v} = (3,1)$

ans. $\underline{u} + \underline{v} = (3,1) + (3,1) = (6,2)$; $\quad |\underline{u}| = |\underline{v}| = \sqrt{10},\ \underline{u} \cdot \underline{v} = 3 \cdot 3 + 1 \cdot 1 = 10$

b) Find the the cosine of the angle between $\underline{v}$ and $\underline{w}$, which of the pairs of vectors are perpendicular (orthogonal)?

1. $\underline{u} = (3,-1), \underline{v} = (4,2)$

ans. $\cos\alpha = \dfrac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{12-2}{\sqrt{10}\sqrt{20}} = \dfrac{10}{\sqrt{200}} = \dfrac{10}{10\sqrt{2}} = \dfrac{1}{\sqrt{2}}, \qquad$ then $\alpha = \dfrac{\pi}{4}$

c). Determine all the vectors which are perpendicular (orthogonal) to the vector $\underline{v}$.

1. $\underline{u} = (1,2)$

We seek $|\underline{v}| = (x,y)$, such that $(1,2) \cdot (x,y) = 0$, $\ x + 2y = 0,\ \ x = -2y,$ then

$|\underline{v}| = (-2y, y)\ , y \in R.$

There are infinitely many such vectors, they are all 'extensions/contractions' of vector $(-2,1)$.

Complex numbers

1. Determine the following:

$b)\ \ Re\left[\dfrac{2+5i}{i-1}\right], \qquad Im\left[\dfrac{2+5i}{i-1}\right]$

$\dfrac{2+5i}{-1+i} \cdot \dfrac{-1-i}{-1-i} = \dfrac{-2-2i-5i-5i^2}{1-i^2} = \dfrac{-2+5-7i}{2}; \quad Re\left[\dfrac{2+5i}{-1+i}\right] = \dfrac{3}{2}\ ;\ Im\ \left[\dfrac{2+5i}{-1+i}\right] = -\dfrac{7}{2}$

$g)$ For real numbers the absolute value can be split up : e.g. $\ |x \cdot (x+2)| = |x| \cdot |x+2|$. The same happens for complex numbers.

$$\left|\frac{i^7(1+i)^8}{\left(\sqrt{2}-i\sqrt{6}\,\right)^{12}}\right| = \frac{|i|^7 \cdot |(1+i)|^{\,8}}{\left|\sqrt{2}-i\sqrt{6}\right|^{12}}$$

We calculate the specific absolute values:

$$|i| = 1$$
$$|1+i| = \sqrt{1+1} = \sqrt{2}$$
$$\left|\sqrt{2}-i\sqrt{6}\right| = \sqrt{2+6} = \sqrt{8}$$

now we see

$$\frac{|i|^7 \cdot |(1+i)|^{\,8}}{\left|\sqrt{2}-i\sqrt{6}\right|^{12}} = \frac{1 \cdot \left(\sqrt{2}\right)^8}{\left|\sqrt{8}\right|^{12}} = \frac{2^4}{8^6}$$

$$h)\ \ Im\left[\frac{i^7}{(2-2i)^4}\right] = \ Im\left[\frac{i^6\ i}{2^4(1-i)^4}\right] = Im\left[\frac{(i^2)^3\ i}{2^4[(1-i)^2]^2}\right] = \frac{1}{2^4}\ Im\left[\frac{-\ i}{[1-2i-1]^2}\right] =$$

$$= \frac{1}{2^4}\ Im\left[\frac{-i}{[-2i]^2}\right] = \frac{1}{2^4}\ Im\left[\frac{-i}{4(-1)}\right] = \frac{1}{2^4}\frac{1}{4} = \frac{1}{2^6}$$

3. Solve the following equations for $z \in C$, it is possible that there *are no solutions* or there are *more than one*.

d) $iz^2 - z + 2i = 0;\qquad \Delta = 1 - 4 \cdot i \cdot 2i = 1 + 8 = 9;$

$z_1 = \frac{1+3}{2i} = \frac{2}{i} = -2i\,;\ \ z_2 = \frac{1-3}{2i} = \frac{-2}{2i} = i\ \ -$ two solutions.

f) $2z + (1+i)\overline{z} = 1 - 3i$

$2(x+iy) + (1+i)(x-iy) = |\text{collect real and imaginary parts}| = (3x+y) + i(x+y)$

$$(3x+y) + i(x+y) = 1 - 3i$$

real and imaginary parts on both sides of equation are equal:.

$\begin{cases} 3x + y = 1 \\ x + y = -3 \end{cases}$ now you have to solve this system $\begin{cases} x = 2 \\ y = -5 \end{cases}$.

h) $(z+\overline{z}) + 2(z-\overline{z}) = 3 + 8i;$ let $z = x + iy$

$(x + iy + x - iy) + 2(x + iy - x + iy) = 3 + 8i$

$2x + 4iy = 3 + 8i$

$\begin{cases} 2x = 3 \\ 4y = 8 \end{cases}\qquad z = \frac{3}{2} + 2i.$

j) $\overline{z - i} = 2z + 1$; let $z = x + iy$

$\overline{x + iy - i} = 2x + 2iy + 1$

$\overline{x + i(y-1)} = (2x+1) + 2iy$

$x - i(y-1) = (2x+1) + iy$

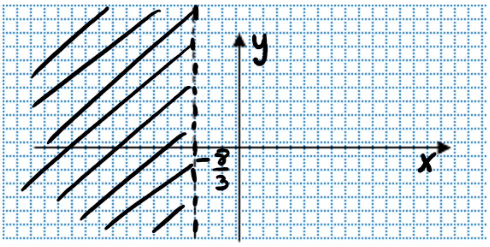
$\begin{cases} x = 2x + 1 \\ -(y-1) = 2y \end{cases}\qquad \begin{cases} x = -1 \\ y = \frac{1}{3} \end{cases}\qquad z = -1 + \frac{1}{3}i$

4. Sketch the following sets in the complex plane

$a)\ \ S = \left\{ z \in C:\ \ Re[z] < Re\left[\frac{-3+2i}{2-i}\right] \right\}$

$$Re\left[\frac{-3+2i}{2-i}\right] = Re\left[\frac{-3+2i}{2-i}\cdot\frac{2+i}{2+i}\right] = Re\left[\frac{-6-3i+4i-2}{4+1}\right] = -\frac{8}{5}$$

$Re[z] = x,\ \ so\ the\ points\ from\ S\ satisfy\ the\ equation\ \ x < -\frac{8}{5}:$



5. Calculate the argument arg (z) and the main argument, $Arg(z)$, of z.

$a)$ $arg(1-i)$, $Arg(1-i)$

$Arg\ (1-i) = -\frac{\pi}{4}$; $\arg(1-i) = -\frac{\pi}{4} + 2k\ \pi$, $k \in I$.

$c)$ $Arg(\sqrt{2} - i\sqrt{6})$

$r = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2}$

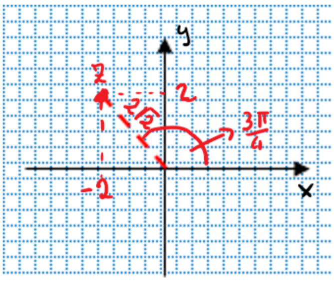
$$\begin{cases} \cos\alpha = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2} \\ \sin\alpha = -\dfrac{\sqrt{6}}{2\sqrt{2}} = -\dfrac{\sqrt{3}}{2} \end{cases} \quad \rightarrow \quad \alpha = -\frac{\pi}{3}$$

$Arg(\sqrt{2} - i\sqrt{6}) = -\frac{\pi}{3}$; $\arg(\ \sqrt{2} - i\sqrt{6}) = -\pi/3 + 2k\pi$

6. Plot the following points and find the polar form (trigonometric form) **and the exponential form** of

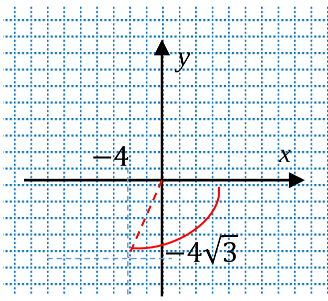
c) $z = -2+2i$; $r = \sqrt{4+4} = 2\sqrt{2}$; $\cos\alpha = -\frac{2}{2\sqrt{2}}$, $\sin\alpha = \frac{2}{2\sqrt{2}} \rightarrow \alpha = 3\frac{\pi}{4}$

$$z = -2+2i = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}\ e^{\frac{3\pi}{4}i}$$



h) $z = -4 - i\ 4\sqrt{3}$; $r = \sqrt{16+48} = 8$; $\cos\alpha = -\frac{4}{8}$, $\sin\alpha = \frac{4\sqrt{3}}{8} \rightarrow \alpha = -\frac{2\pi}{3}$

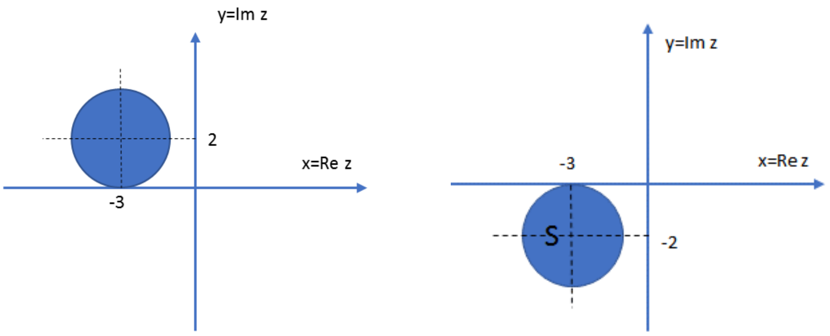
$$z = -4 - i\ 4\sqrt{3} = 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 8\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right) = 8\ e^{\frac{-2\pi}{3}i}$$



7. Sketch the following sets in the complex plane, mark the main points (wedges and circles)

c) $S = \{z \in C : |\overline{z} + 3 - 2i| \leq 2\}$

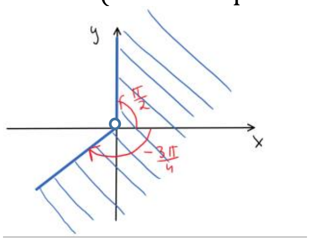
$|\overline{z} + 3 - 2i| \leq 2$ first we can plot the points $\overline{z}$, next take a mirror projection about axis OX of these points:



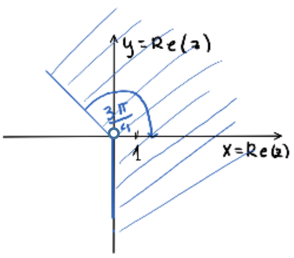
or transform the inequality :

$$|\overline{z} + 3 - 2i| \leq 2$$
$$|\overline{z} + \overline{3 + 2i}| \leq 2$$
$$|\overline{z + 3 + 2i}| \leq 2$$
$$|z + 3 + 2i| \leq 2 \quad inside\ of\ circle\ , centre\ \ c(-3, -2), radius\ \ r \leq 2.$$

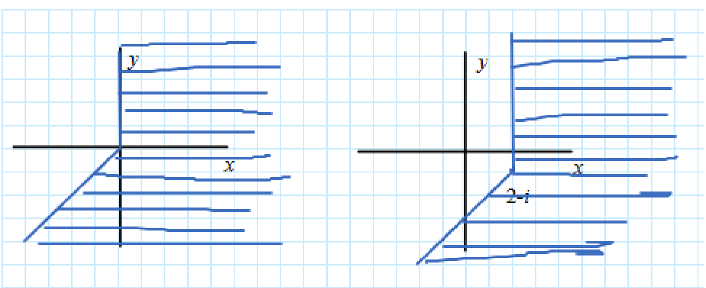
$$f)\ \ S = \left\{z \in C : -\frac{3\pi}{4} \leq arg(z) \leq \frac{\pi}{2}\right\}$$



g) the set from f) is the set of points $\overline{z}$, so now, to obtain S, we should mirror this set with respect to the OX axis



h) First plot the points $z_1 = z - 2 + i$, next to obtain S, points z are shifted by $2 - i$:



$m)\ \ S = \left\{ z \in C\!:\ 0 \le arg\left(\frac{z}{i}\right) \le Arg(3+3i) \right\}$

$Arg(i\,) = \frac{\pi}{2}$ and $Arg(3+3i) = Arg(1+i) = \frac{\pi}{4}$, so

$$0 \le Arg\left(\frac{z}{i}\right) \le Arg(3+3i)$$

$$0 \le \alpha - \frac{\pi}{2} \le \frac{\pi}{4} \quad add \quad \frac{\pi}{2} \ \ to\ both\ sides.$$

$$\frac{\pi}{2} \le \alpha \le \frac{3\pi}{4}$$

Plot a wedge.

$n)\quad S = \{z \in C\!:\ Arg(1-3i) \le arg\, z \le Arg(-2+5i)\}$

Simply plot the numbers $z_1 = 1-3i$ oraz $z_2 = -2+5i$ *i.e.* points $P_1(1,-3)$, $P_2(-2{,}5)$ on the plane, the region is an infinite wedge between the line passing through points (0,0), $P_1(1,-3)$ and the line passing through (0,0), $P_2(-2{,}5)$.

$m)\ S = \{\ z \in C\!:\ \mathrm{Im}[(2+i)(3+5i)] \ge \left|z - \overline{3+i}\right| \ge \left|\sqrt{5} + 2i\,\right|$

$\wedge\quad \mathrm{Arg}(3-i)\ \le \arg(z) \le \mathrm{Arg}\left[e^{\,i\frac{\pi}{2}}\right]\ \}$

1. First we calculate the values on the left- and right-hand sides of $\mathrm{Im}[(2+i)(3+5i)] \ge$ $\left|z - \overline{3+i}\right| \ge \left|\sqrt{5} + 2i\,\right|$

$1a.\ \mathrm{Im}[(2+i)(3+5i)] = Im\,[6+10i+3i+\ 5i^2] = 13$

$1b.\ \left|\sqrt{5}+2i\,\right| = \sqrt{\left(\sqrt{5}\right)^2 + 2^2} = \sqrt{5+4} = 3$

2. The "in-between"

$2\,.\left|z - \overline{3+i}\right| = |z-3-i| = |z-(3+i)|$ *the distance between some* 'z" *and* $z_0 = 3+i$

$2.\ \mathrm{Im}[(2+i)(3+5i)] \ge \left|z - \overline{3+i}\right| \ge \left|\sqrt{5}+2i\,\right|$ *means*
$13 \ge |z-(3+i)| \ge 3$ this is a ring with centre at P(3,1) and radii between 3, 13.

3. The second inequality:
$3.\ \mathrm{Arg}(3-i)\ \le \arg(z) \le \mathrm{Arg}\left[e^{\,i\frac{\pi}{2}}\right]$
$3a.\ \mathrm{Arg}(3-i)$ angle between the positive OX axis and the point P(3,-1), we don not calculate the argument – ONLY plot it.
$3b.\ \mathrm{Arg}\left[e^{\,i\frac{\pi}{2}}\right] = \frac{\pi}{2} -$ simple

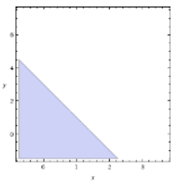
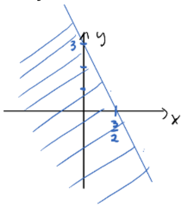
so $\mathrm{Arg}(3-i)\ \le \arg(z) \le \mathrm{Arg}\left[e^{\,i\frac{\pi}{2}}\right]$ is a infinite wedge between line P(0,0) to P(3,-1), and line P(0,0) to P(0,1).

The intersection of both sets is S, top, right part of ring.

8. Sketch the following sets in the complex plane, mark the main points

b) $S = \{z \in C\!: Im\,[(1+2i)z - 3i] < 0\}$,

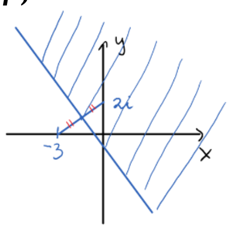
$ans.:$ let $z = x + iy$, then
$Im\,[(1+2i)z - 3i] = Im\,[(1+2i)(x+iy) - 3i] = Im\,[(x-2y) + i(2x+y-3)]$:
$2x + y - 3 < 0 \;\Leftrightarrow\; y < -2x + 3.$

d) $S = \{z \in C\colon \overline{z+i} = z - 1\}$,
$\overline{z+i} = z - 1$
$\overline{z} - i = z - 1$
$\overline{z} - z = i - 1$, let $z = x + iy$, then since the real part are equal and the imaginary parts are equal:

$\overline{x+iy} - (x+iy) = i - 1$
$x - iy - (x+iy) = i - 1$
$-2iy = i - 1; \qquad S = \emptyset.$

f)



g) $S = \{z \in C\colon |iz + 1 - i| < 2\}$,
$|iz + 1 - i| = |i| \cdot \left|z + \dfrac{1}{i} - 1\right| = |i| \cdot |z - i - 1| = |z - (i+1)| < 2;$
inside of circle, centre $c(1,1)$, radius $r = 2$. Sketch it.

h) $S = \{z \in C\colon |\overline{z - i + 1}| < 3\}$:
$|\overline{z - i + 1}| = |z - i + 1\,| = |z - (-1 + i)\,| < 3$; inside of circle, centre c(-1,1) i radius r = 3. Sketch it.

i) $S = \{z \in C\colon |z - 2i| + |z + 2i| = 4\,\}$
The set consists of complex numbers which lie in distance equal to 4 from points -2i and 2i. This is an interval on the imaginary axis with from -2i to 2i.

j) $S = \{z \in C\colon |\bar{z} + i| < 2\}$
Method 1: $|\bar{z} + i| = |\bar{z} - \bar{i}| = |\overline{z - i}\,| = |z - i| < 2$

or Method 2: $|\overline{x + iy} + i| = |x - iy + i| = |x + i(-y+1) = \sqrt{x^2 + (y-1)^2} < 2.$
Inside of circle, centre c(0,1), radius r = 2. Sketch it

9. Find Arg(z), |z| for the following complex numbers

$a)\;\; z = \left(e^{\frac{i\pi}{5}}\right)^{15} \quad arg(z) = 15\dfrac{\pi}{5} = 3\pi : Arg(z) = \pi; \;\; |z| = 1$

$b)\;\; z = (1+i)^3 \cdot e^{\frac{i\pi}{4}} = \left(\sqrt{2}\;\; e^{\frac{i\pi}{4}}\right)^3 \cdot e^{\frac{i\pi}{4}} = \;2\sqrt{2}\;\; e^{\frac{i\,3\pi}{4} + \frac{i\,\pi}{4}} \;=\; 2\sqrt{2}\;\; e^{i\,\pi}\;:$

$Arg(z) = \pi; \;\; |z| = 2\sqrt{2}$

$$c)\; z = \frac{(-3+3i)^{10}}{\left(e^{\frac{i\pi}{3}}\right)^4} = \frac{\left(3\sqrt{2}\, e^{\frac{3}{4}\pi i}\right)^{10}}{\left(e^{\frac{i\pi}{3}}\right)^4} = \left(3\sqrt{2}\right)^{10} e^{\frac{30}{4}\pi i - \frac{4}{3}\pi i} = 3^{10} 2^5 \; e^{\frac{37}{6}\pi i};$$

$$\frac{37}{6}\pi = \left(\frac{36}{6} + \frac{1}{6}\right)\pi = \frac{\pi}{6} + 2\cdot 3\pi; \; : Arg(z) = \frac{\pi}{6}; \;\; |z| = 3^{10}2^5$$

10. Let $z = -1 + i$, write the following complex numbers in exponential form

First write z in exponential form: $z = -1 + i = \sqrt{2}\, e^{\frac{3\pi}{4} i}$;

$$a) \quad -z = -1\cdot\sqrt{2}\, e^{\frac{3\pi}{4} i} = e^{\pi i}\sqrt{2}\, e^{\frac{3\pi}{4} i} = \sqrt{2}\, e^{\frac{7\pi}{4} i} = \sqrt{2}\, e^{-\frac{\pi}{4} i}$$

$$b) \quad iz = e^{\frac{\pi}{2} i}\sqrt{2}\, e^{\frac{3\pi}{4} i} = \sqrt{2}\, e^{\frac{5\pi}{4} i} = \sqrt{2}\, e^{-\frac{3\pi}{4} i}$$

$$c) \quad \frac{1}{z} = \frac{1}{\sqrt{2}\, e^{\frac{3\pi}{4} i}} = \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4} i}$$

11*. Let $z = 2\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$, write in exponential form

First write z in exponential form $z = 2\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right) = 2\, e^{\frac{\pi}{7} i}$

$$a) -z = -1\cdot\; 2\, e^{\frac{\pi i}{7}} = 2\, e^{\pi i}\, e^{\frac{\pi}{7} i} = 2\;\; e^{\frac{8\pi i}{7}} = |\, main\ argument\,| = 2e^{\frac{-6\pi i}{7}}$$

$$b)\; iz = i\cdot\; 2\, e^{\frac{\pi i}{7}} = 2\, e^{\frac{\pi}{2} i}\, e^{\frac{\pi}{7} i} = 2\;\; e^{\frac{9\pi i}{14}}$$

$$c) \frac{1}{z} = \frac{1}{2\, e^{\frac{\pi}{7} i}} = \frac{1}{2}\;\; e^{-\frac{\pi}{7} i}$$

$$d)\; \bar{z} = 2\, e^{-\frac{\pi}{7} i}$$

$$e)\; \left(1 + i\sqrt{3}\right) z = \left(1 + i\sqrt{3}\right)\cdot\; 2\, e^{\frac{\pi i}{7}} = 2\, e^{\frac{\pi}{3} i}\; 2\, e^{\frac{\pi}{7} i} = 4\;\; e^{\frac{10\pi i}{21}}$$

$$f)\; z^{10} = 2^{10}\, e^{\frac{\pi i}{7}\cdot 10} = 2^{10}\;\; e^{\frac{10\pi}{7} i} = |main\ argument\,| = 2e^{\frac{-4\pi}{7} i}$$

12. First express the complex number z in exponential, and polar form, next express it in algebraical/canonical form $z = x + iy$.

$$\boldsymbol{c})\;\; z = \frac{(1+i)^{22}}{\left(1 - i\sqrt{3}\right)^6} = \frac{\left(\sqrt{2}\,(e^{i\,\pi/4}\right)^{22}}{(2\, e^{-i\,\pi/3})^6} = 2^5\, e^{\frac{i\,22\pi}{4} - \left(-\frac{i6\pi}{3}\right)} = 2^5\, e^{-\frac{\pi}{2} i} = 2^5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$
$$= 0 - 32i$$

13*. Calculate the Cartesian coordinates of the point Q obtained by rotating point $P(2,3)$ by $60°$ around (0,0) (hint: use the multiplication of complex numbers).

The point obtained by rotation is $P' = (2+3i)\cdot e^{\frac{i\,\pi}{3}} = (2+3i)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) =$

$(2+3i)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$, so $P'\left(1-\frac{3\sqrt{3}}{2}, \sqrt{3}+\frac{3}{2}\right)$.

14. Calculate and plot in the complex plane, the real and imaginary parts of the indicated complex numbers, ***remember there might be more than one value.*** Where possible find the algebraic values of the coordinates

$f)\ \sqrt{2\sqrt{3}-2i}$; represent the number in exponential form

$2\sqrt{3}-2i;\ r=4, \cos\alpha=\frac{\sqrt{3}}{2}, \qquad \sin\alpha=-\frac{1}{2} \ \Leftrightarrow \alpha=-\frac{\pi}{6}$

$2\sqrt{3}-2i=4e^{-\frac{\pi}{6}i}$

$\sqrt{2\sqrt{3}-2i}\in\{z_0,z_1\}; \quad z_0=2e^{-\frac{\pi}{12}i}, \qquad z_0=2e^{\frac{11}{12}\pi i}$

$g)\ \sqrt{5+12i}$
$(x+iy)^2=5+12i$
$x^2-y^2+2xyi=5+12i$ two equations are obtained:

$\begin{cases} x^2-y^2=5 \\ 2xy=12 \end{cases}$ skąd $x=\frac{6}{y};\ \ y^4+5y^2-36=0$ so $z_0=3+2i, \ \ z_1=-3-2i$.

$\sqrt{5+12i}\in\{3+2i,-3-2i\}$

$h)\ \sqrt{8+6i}$
Take $z=x+iy$, then $z^2=8+6i$ and solve for x, y

15. Solve for z:

$\boldsymbol{c})\ z(1+i)^2=1,$
$z=\frac{1}{(1+i)^2}=\frac{1}{2i}=-\frac{i}{2}$

$\boldsymbol{e})\ \frac{i}{z^3}-\frac{1}{27i}=0$
$\frac{i}{z^3}=\frac{1}{27i} \quad\Leftrightarrow\quad z^3=-27 \quad\Leftrightarrow\quad z=\sqrt[3]{-27}, \quad z\in\left\{3\ e^{i\frac{\pi}{3}},\ 3\ e^{i\pi},\ 3\ e^{i\frac{5\pi}{3}}\right\}$

$\boldsymbol{f})\ \frac{z^4}{i+1}=\sqrt{2}\,e^{i\frac{\pi}{4}}$

$\frac{z^4}{i+1}=\sqrt{2}\,e^{i\frac{\pi}{4}}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=1+i$

thus: $\frac{z^4}{i+1}=1+i$

$z^4=(1+i)(1+i)=1+2i+i^2=2i$

$z=\sqrt[4]{2i}$

We need the n-th root formula:

$$\sqrt[n]{z}=\{z_0,z_1,z_2,\dots,z_k\};$$

$$z_k = \sqrt[n]{r}\ \left(\cos\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right)\right) = \sqrt[n]{r}\cdot e^{i\left(\frac{\alpha}{n}+k\frac{2\pi}{n}\right)}$$

First we have to express $2i$ in exponential form: $2i = 2e^{\frac{i\pi}{2}}$, i.e. $\alpha = \frac{\pi}{2},\ r = 2.$

$$\sqrt[4]{z} = \{z_0, z_1, z_2, z_3\};\quad z_k = \sqrt[4]{2}\cdot e^{i\left(\frac{\pi/2}{4}+k\frac{2\pi}{4}\right)}$$

So

$z_0 = \sqrt[4]{2}\cdot e^{i\left(\frac{\frac{\pi}{2}}{4}+0\frac{2\pi}{4}\right)} = \sqrt[4]{2}\cdot e^{\frac{i\pi}{8}}\ (in\ Arg)$

$z_1 = \sqrt[4]{2}\cdot e^{i\left(\frac{\frac{\pi}{2}}{4}+1\,\frac{2\pi}{4}\right)} = \sqrt[4]{2}\cdot e^{i\left(\frac{\pi}{8}+\frac{\pi}{2}\right)} = \sqrt[4]{2}\cdot e^{i\left(\frac{5\pi}{8}\right)}\ (in\ Arg)$

$z_2 = \sqrt[4]{2}\cdot e^{i\left(\frac{\frac{\pi}{2}}{4}+2\,\frac{2\pi}{4}\right)} = \sqrt[4]{2}\cdot e^{i\left(\frac{9\pi}{8}\right)} = (in\ Arg) = \sqrt[4]{2}\cdot e^{i\,\frac{-7\pi}{8}}.$

$z_3 = \sqrt[4]{2}\cdot e^{i\left(\frac{\frac{\pi}{2}}{4}+3\,\frac{2\pi}{4}\right)} = \sqrt[4]{2}\cdot e^{i\left(\frac{13\pi}{8}\right)} = (in\ Arg) = \sqrt[4]{2}\cdot e^{i\frac{-3\pi}{8}}.$

We sketch 4 vertices of a square: z_0, z_1, z_2, z_3.

16. Let $z_1 = 3i + i^2,\ z_2 = \frac{2}{1-i}$

e) give the geometric interpretation of the above operations (sum, product, cubic root, power) and plot the results.

ans to *e*) :

a) when adding two complex numbers we add corresponding two vectors with beginning at 0 and end points at these numbers;

b) all the *n-th* roots of a complex number z, lie on a circle of centre 0 and of radius $\sqrt[n]{|z|}$. Each one of them is obtained by rotating a chosen root (e.g. one with the argument $\frac{1}{n}\ Arg\ z$) by the angle equal to multiplicity of $2\frac{\pi}{n}$. The roots are the vertices of a regular *n*-gon .

c) when multipying complex numbers we multiply the absolute values and add the arguments

d) when taking the *n*-the power of a complex number we the *n*-the power of the absolute value and the *n*-th multiplicty of the argument.

17*. Use the de Moivre'a Formula to determine the dependence of $\sin 2\alpha$ and $\cos 2\alpha$ on the $\sin\alpha$ and $\cos\alpha$ (i.e. formulas for $\sin 2\alpha$ and $\cos 2\alpha$ which contain only '$\sin\alpha$' and '$\cos\alpha$').

Let $z = \cos\alpha + i\sin\alpha$, then form the de Moivre'a Formula: $z^2 = \cos 2\alpha + i\sin 2\alpha$

On the other hand, if we square both sides of the equation $z = \cos\alpha + i\sin\alpha$, then

$z^2 = (\cos\alpha + i\sin\alpha)^2 = \cos^2\alpha - \sin^2\alpha + 2i\sin\alpha\cos\alpha$, thus

$\cos 2\alpha + i\sin 2\alpha = \cos^2\alpha - \sin^2\alpha + 2i\sin\alpha\cos\alpha$, real and imaginary parts should be equal:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha\ \ ;\ \ \sin 2\alpha = 2\sin\alpha\ \cos\alpha\ .$$

18*. Use the exponential form to solve

b) $\frac{|z|^2\, z}{\bar{z}^3} = -1,\ z \neq 0$ i niech $z = |z|e^{i\,\alpha}$. then the equation becomes:

$$\frac{|z|^2\ |z|e^{i\,\alpha}}{\overline{|z|e^{i\,\alpha}}^{3}} = -1$$

$$\frac{|z|^2\,|z|e^{i\,\alpha}}{|z|^3\,e^{-i3\,\alpha}} = -1 \;\Leftrightarrow\; e^{i4\,\alpha} = -1 \quad \Leftrightarrow\; e^{i4\,\alpha} = e^{i(\pi+2k\pi)} \;\Leftrightarrow\; 4\,\alpha = \pi + 2k\pi$$ and finally

we obtain z number with an arbitrary $|z|$a nd four different arguments:

$$\alpha = \frac{\pi}{4} + \frac{k\pi}{2} \quad k = 0,1,2,3.$$

19*. Write $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ in exponential form

a) calculate all the possible integer powers $z^n,\ n \in I,\ \ (I = \text{Integers})$

First we express $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ in exponential form: $z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i(\frac{\pi}{3}+2k\pi)}$, so $z^n = e^{i\,n\frac{\pi}{3}}$,

($n \cdot 2k\pi$ is a multiplicity of the period $2k\pi$). The periodicity of sine and cosine causes that there are only 6 different values of the powers of n i.e. for $n = 6k, 6k+1, 6k+2, \dots, 6k+5,\ k \in I$ so

$$z^n = e^{i\,n\frac{\pi}{3}} = \begin{cases} 1 & for \quad n = 6k \\ \frac{1}{2} + \frac{i\sqrt{3}}{2} & for \quad n = 6k+1 \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & for \quad n = 6k+2 \\ -1 & for \quad n = 6k+3 \\ -\frac{1}{2} - \frac{i\sqrt{3}}{2} & for \quad n = 6k+4 \\ \frac{1}{2} - \frac{i\sqrt{3}}{2} & for \quad n = 6k+5 \end{cases}$$

b) z^i, where i is the imaginary unit $i^2 = -1$.

Express $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ in exponential form; $z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i(\frac{\pi}{3}+2k\pi)}$, $k \in I$,

So $z^i = e^{i\cdot i\,(\frac{\pi}{3}+2k\pi)} = e^{-\,(\frac{\pi}{3}+2k\pi)}$. This means that $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^i$ are infinitely many <u>real numbers</u>. .

20 . Calculate the power e^i, where i is the imaginary unit $i^2 = -1$.

$e^i = \cos 1 + i \sin 1$.

21. Find all the complex roots of the equations

a) According to the Rational Root Test, it is easily seen that -1 is a root of the polynomial. We divide the polynomial by (x + 1) to obtain that $x^3 - x^2 + 3x + 5 = (x+1)(x^2 - 2x + 5)$, next we find the roots of the quadratic polynomial $x^2 - 2x + 5$.

b) $2z^3 + 4z^2 + 3z + 6 = (z+2)(2z^3+3)$, roots $z_1 = i\sqrt{\frac{3}{2}}, z_2 = -i\sqrt{\frac{3}{2}}$, $z_3 = -2$.

d) $z^3 - \frac{7}{6}z^2 - \frac{3}{2}z - \frac{1}{3} = 0 \quad | \cdot 6 \;\Leftrightarrow\; 6z^3 - 7z^2 - 9z - 2 = 0$

$6z^3 - 7z^2 - 9z - 2 = (z-2)(6z^2 + 5z + 1)$

22. Let

$b)$ $z_1 = -i\sqrt{2},\ z_2 = i$ be two of the roots of $z^6 - 2z^5 + 5z^4 - 6z^3 + 8z^2 - 4z + 4 = 0$ find all the other roots.

$z^6 - 2z^5 + 5z^4 - 6z^3 + 8z^2 - 4z + 4 = (z + i\sqrt{2})(z - i\sqrt{2})(z - 1)(z + i)(z^2 - 2z + 2)$
The equation $z^2 - 2z + 2$ has the following two roots $z_5 = 1 + i,\ \ z_6 = 1 - i$.

23. Write a polynomial with real coefficients of the fourth degree which has the following roots: $z_1 = 1 - i,\ z_2 = 3i.$

$$\text{e.g. } \big(z - (1 - i)\big)\big(z - (1 + i)\big)(z - 3i)(z + 3i) = (z^2 - 2z + 2)(z^2 + 9)$$
$$= z^4 - 2z^3 + 11z^2 - 18z + 18.$$